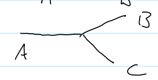
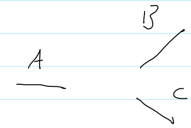
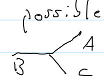
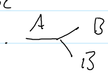
The Feynman Rules (or how to calculate $\mathcal{M}$)

A very simple "toy" model:

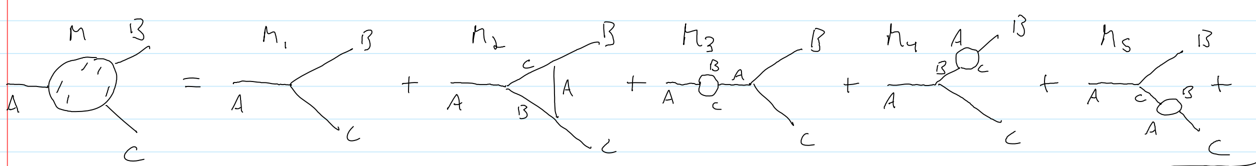
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A - \frac{1}{2}\left(\frac{m_A c}{\hbar}\right)^2\phi_A^2 + \frac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \frac{1}{2}\left(\frac{m_B c}{\hbar}\right)^2\phi_B^2 + \frac{1}{2}\partial_\mu\phi_C\partial^\mu\phi_C - \frac{1}{2}\left(\frac{m_C c}{\hbar}\right)^2\phi_C^2 - g\,\phi_A\phi_B\phi_C$$

1) 3 real spin-0 particles A, B, C
2) They are their own antiparticles, e.g. $A = \bar{A}$ or $\overset{A}{\text{———}}$ ← no direction
3) $m_A > m_B + m_C$
4)  basic interaction vertex

The first step is to draw the diagram(s). Beginning w/ the initial and final states, e.g.  we then "connect" these in all possible ways using the interaction vertices. Note that we can rotate the vertices, e.g. , but we cannot change the particle content, e.g.  is not allowed.

At first it might seem obvious how to connect the pieces, but we quickly realize there are many complicated ways.

Example: "Decay of A into B+C"



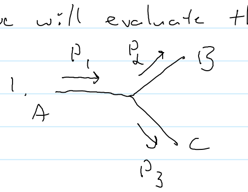
Tree-level — 1st order

Vertex-correction, self-energy-corrections — 3rd order

We evaluate each diagram to get M_i using these rules (will change somewhat for full SM):

1. Label all momenta p_i - external, q_i - internal w/ arrows next to lines. This lets us keep track of momentum flow (different from particle identity flow). For p_i the arrows must go forward in time, but for q_i it doesn't matter.
2. For each vertex write a factor of $-ig$ (g is the coupling strength —<).
3. For each internal line we write a factor $i/(q_j^2 - m_j^2c^2)$ Note: $q_j^2 \neq m_j^2c^2$ since virtual
4. For each vertex conserve 4-momentum w/ $(2\pi)^4\delta^4(P_{tot,in} - P_{tot,out})$ where $P = \sum p_i, q_i$
5. Integrate everything you have written over all internal 4-momenta w/ "factors" $\int \frac{1}{(2\pi)^4} d^4q_j$
6. After this you will have an overall $(2\pi)^4\delta^4(p_{tot,in} - p_{tot,out})$. Erase this and multiply by i to get $\mathcal{M}$.

As a trivial example we will evaluate the tree-level diagram above one step at a time:

1. 

2. $-ig$

3. skip since no internal lines

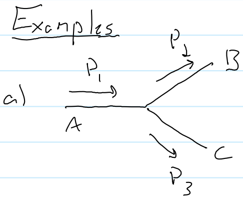
4. $-ig(2\pi)^4\delta^4(p_1 - p_2 - p_3)$

5. skip since no internal lines

6. $-ig\cancel{(2\pi)^4\delta^4(p_1 - p_2 - p_3)} \times i = g$

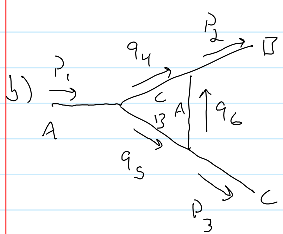
Examples

a)



$$-ig(2\pi)^4\delta^4(p_1-p_2-p_3) \Rightarrow \mathcal{M}=g$$

b)



$$\iiint (-ig)^3 \frac{i}{q_4^2-m_C^2c^2}\frac{i}{q_5^2-m_B^2c^2}\frac{i}{q_6^2-m_A^2c^2}(2\pi)^4\delta^4(p_1-q_4-q_5)(2\pi)^4\delta^4(q_4+q_6-p_2)$$
$$\times(2\pi)^4\delta^4(q_5-q_6-p_3)\frac{d^4q_4}{(2\pi)^4}\frac{d^4q_5}{(2\pi)^4}\frac{d^4q_6}{(2\pi)^4}$$

Use $\delta^4(p_1-q_4-q_5)$ to do q_4 integral (replacing $q_4=p_1-q_5$ everywhere):

$$\iint (-ig)^3 \frac{i}{(p_1-q_5)^2-m_C^2c^2}\frac{i}{q_5^2-m_B^2c^2}\frac{i}{q_6^2-m_A^2c^2}(2\pi)^4\delta^4(p_1-q_5+q_6-p_2)$$
$$(2\pi)^4\delta^4(q_5-q_6-p_3)\frac{d^4q_5}{(2\pi)^4}\frac{d^4q_6}{(2\pi)^4}$$

Use $\delta^4(q_5-q_6-p_3)$ to do q_6 integral:

$$\int (-ig)^3 \frac{i}{(p_1-q_5)^2-m_C^2c^2}\frac{i}{q_5^2-m_B^2c^2}\frac{i}{(q_5-p_3)^2-m_A^2c^2}(2\pi)^4\delta^4(p_1-p_2-p_3)\frac{d^4q_5}{(2\pi)^4}$$

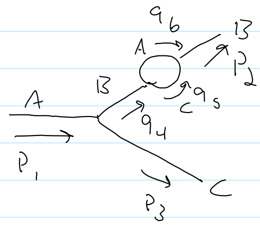
or

$$(2\pi)^4\delta^4(p_1-p_2-p_3)(-ig)^3i^3\int\frac{1}{(p_1-q_5)^2-m_C^2c^2}\frac{1}{q_5^2-m_B^2c^2}\frac{1}{(q_5-p_3)^2-m_A^2c^2}\frac{d^4q_5}{(2\pi)^4}$$

Then:

$$\mathcal{M}=i(-ig)^3i^3\int\frac{1}{(p_1-q_5)^2-m_C^2c^2}\frac{1}{q_5^2-m_B^2c^2}\frac{1}{(q_5-p_3)^2-m_A^2c^2}\frac{d^4q_5}{(2\pi)^4}$$

c)



$$\int\int\int (-ig)^3 \frac{i}{q_4^2 - m_B^2c^2}\,\frac{i}{q_5^2 - m_C^2c^2}\,\frac{i}{q_6^2 - m_A^2c^2}\,(2\pi)^4\delta^4(p_1 - q_4 - p_3)$$

$$\times (2\pi)^4\delta^4(q_4 - q_5 - q_6)$$

$$\times (2\pi)^4\delta^4(q_5 + q_6 - p_2)$$

$$\times \frac{d^4q_4}{(2\pi)^4}\,\frac{d^4q_5}{(2\pi)^4}\,\frac{d^4q_6}{(2\pi)^4}$$

Use $\delta(q_5+q_6-p_2)$ to do $\int d^4q_6$:

$$\int\int (-ig)^3 \frac{i}{q_4^2 - m_B^2c^2}\,\frac{i}{q_5^2 - m_C^2c^2}\,\frac{i}{(p_2-q_5)^2 - m_A^2c^2}\,(2\pi)^4\delta^4(p_1 - q_4 - p_3)$$

$$\times (2\pi)^4\delta^4(q_4 - q_5 - p_2 + q_5)$$

$$\times \frac{d^4q_4}{(2\pi)^4}\,\frac{d^4q_5}{(2\pi)^4}$$

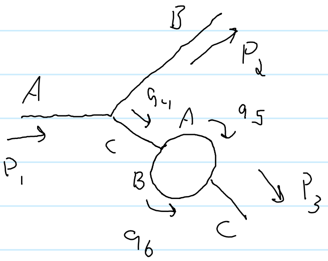
Use $\delta(q_4-p_2)$ to do $\int d^4q_4$:

$$\int (-ig)^3 \frac{i}{p_2^2 - m_B^2c^2}\,\frac{i}{q_5^2 - m_C^2c^2}\,\frac{i}{(p_2-q_5)^2 - m_A^2c^2}\,(2\pi)^4\delta^4(p_1 - p_2 - p_3)\,\frac{d^4q_5}{(2\pi)^4}$$

Then:

$$M = i(-ig)^3 i^3 \,\frac{1}{p_2^2 - m_B^2c^2}\int \frac{1}{q^2 - m_C^2c^2}\,\frac{1}{(p_2-q)^2 - m_A^2c^2}\,\frac{d^4q}{(2\pi)^4}$$

d)



$$\iiint (-ig)^3 \frac{i}{q_4^2 - m_C^2c^2}\,\frac{i}{q_5^2-m_A^2c^2}\,\frac{i}{q_6^2-m_B^2c^2}\,(2\pi)^4\delta^4(p_1-q_4-p_2)\,(2\pi)^4\delta^4(q_4-q_5-q_6)\,(2\pi)^4\delta^4(q_5+q_6-p_3)\,\frac{d^4q_4}{(2\pi)^4}\,\frac{d^4q_5}{(2\pi)^4}\,\frac{d^4q_6}{(2\pi)^4}$$

Use $q_5 = p_3 - q_6$ to do q_5 integral

$$\iint (-ig)^3 \frac{i}{q_4^2-m_C^2c^2}\,\frac{i}{(p_3-q_6)^2-m_A^2c^2}\,\frac{i}{q_6^2-m_B^2c^2}\,(2\pi)^4\delta^4(p_1-q_4-p_2)\,(2\pi)^4\delta^4(q_4-p_3+\cancel{q_6}-\cancel{q_6})\,\frac{d^4q_4}{(2\pi)^4}\,\frac{d^4q_6}{(2\pi)^4}$$

Use $q_4 = p_1 - p_2$ to do q_4

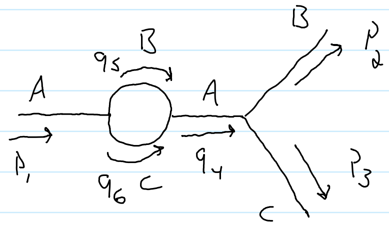
$$\int (-ig)^3 \frac{i}{(p_1-p_2)^2-m_C^2c^2}\,\frac{i}{(p_3-q_6)^2-m_A^2c^2}\,\frac{i}{q_6^2-m_B^2c^2}\,(2\pi)^4\delta^4(p_1-p_2-p_3)\,\frac{d^4q_6}{(2\pi)^4}$$

$$\mathcal{M} = i(-ig)^3\, i^3 \,\frac{1}{(p_1-p_2)^2-m_C^2c^2}\int \frac{1}{(p_3-q_6)^2-m_A^2c^2}\,\frac{1}{q_6^2-m_B^2c^2}\,\frac{d^4q_6}{(2\pi)^4}$$

Since $p_1 = p_2 + p_3 \Rightarrow p_1 - p_2 = p_3$ so this only depends on p_3.

$$\mathcal{M} = i(-ig)^3\, i^3 \,\frac{1}{p_3^2-m_C^2c^2}\int \frac{1}{(p_3-q)^2-m_A^2c^2}\,\frac{1}{q^2-m_B^2c^2}\,\frac{d^4q}{(2\pi)^4}$$

e)



$$\iiint (-ig)^3 \frac{i}{q_4^2 - m_A^2c^2} \frac{i}{q_5^2 - m_B^2c^2} \frac{i}{q_6^2 - m_C^2c^2} (2\pi)^4 \delta^4(p_1 - q_5 - q_6) (2\pi)^4 \delta^4(q_5 + q_6 - q_4) (2\pi)^4 \delta^4(q_4 - p_2 - p_3) \frac{d^4q_4}{(2\pi)^4} \frac{d^4q_5}{(2\pi)^4} \frac{d^4q_6}{(2\pi)^4}$$

Use $q_5 = p_1 - q_6$ to do q_5 integral

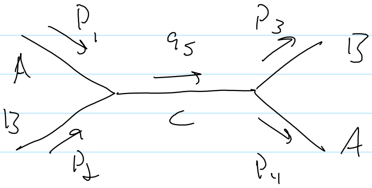
$$\iint (-ig)^3 \frac{i}{q_4^2 - m_A^2c^2} \frac{i}{(p_1 - q_6)^2 - m_B^2c^2} \frac{i}{q_6^2 - m_C^2c^2} (2\pi)^4 \delta^4(p_1 - \cancel{q_6} + \cancel{q_6} - q_4) (2\pi)^4 \delta^4(q_4 - p_2 - p_3) \frac{d^4q_4}{(2\pi)^4} \frac{d^4q_6}{(2\pi)^4}$$

Use $q_4 = p_1$ to do q_4 integral

$$\int (-ig)^3 \frac{i}{p_1^2 - m_A^2c^2} \frac{i}{(p_1 - q)^2 - m_B^2c^2} \frac{i}{q^2 - m_C^2c^2} (2\pi)^4 \delta(p_1 - p_2 - p_3) \frac{d^4q}{(2\pi)^4}$$

$$M = i(-ig)^3 i^3 \frac{1}{p_1^2 - m_A^2c^2} \int \frac{1}{(p_1 - q)^2 - m_B^2c^2} \frac{1}{q^2 - m_C^2c^2} \frac{d^4q}{(2\pi)^4}$$

As another example consider the scattering process $A+B \to A+B$ at lowest order:

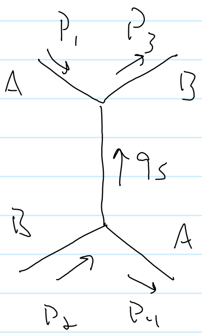


$$\int (-ig)^2 \frac{i}{q_5^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1 + p_2 - q_5)\, \underbrace{(2\pi)^4 \delta^4(q_5 - p_3 - p_4)}_{\text{Use to set } q_5 = p_3 + p_4} \frac{d^4 q_5}{(2\pi)^4}$$

$$(-ig)^2\, i \frac{1}{(p_3+p_4)^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)$$

Then:

$$\mathcal{M}_1 = \frac{g^2}{(p_3+p_4)^2 - m_C^2 c^2}$$



$$\int (-ig)^2 \frac{i}{q_5^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1 + q_5 - p_3) (2\pi)^4 \delta^4(p_2 - q_5 - p_4) \frac{d^4 q_5}{(2\pi)^4}$$

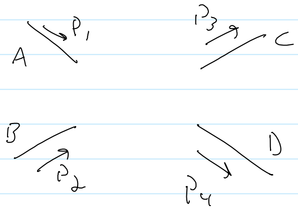
$$(-ig)^2\, i \frac{1}{(p_2-p_4)^2 - m_C^2 c^2} (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)$$

Then:

$$\mathcal{M}_2 = \frac{g^2}{(p_2-p_4)^2 - m_C^2 c^2}$$

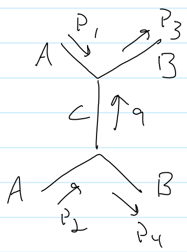
So to leading order we have $\mathcal{M} = g^2 \left[\frac{1}{(p_3+p_4)^2 - m_C^2 c^2} + \frac{1}{(p_2-p_4)^2 - m_C^2 c^2} \right]$

When evaluating various Feynman diagrams (especially when you plan to add several contributions) it helps to start with all of the external labels:

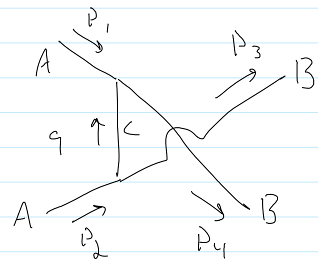


Then fill in the rest of the diagram. This way you can avoid mistakenly using a momentum p_i for different particles in different diagrams. It is important when adding M_i's that a single external momentum label, e.g. p_1, always refer to the same particle.

What about



vs.



Are they different? Should they both be included?

They are definitely different!

For example if
$p_1 = 100$
$p_2 = 2$
$p_3 = 101$
$p_4 = 1$

Then for the first diagram $q = 1$

while for the second diagram $q = -99$

Both should be included as long as they are allowed by the rules of the theory!

$$\int (-ig)^2 \frac{i}{q^2 - m_c^2 c^2} (2\pi)^4 \delta^4(p_1 + q - p_3) (2\pi)^4 \delta^4(p_2 - q - p_4) \frac{d^4 q}{(2\pi)^4}$$

$$\Downarrow$$

$$(-ig)^2 \frac{i}{(p_3 - p_1) - m_c^2 c^2} (2\pi)^4 \delta^4(p_2 - p_3 + p_1 - p_4)$$

$$\Downarrow$$

$$M = \frac{g^2}{(p_3 - p_1) - m_c^2 c^2}$$

$$\int (-ig)^2 \frac{i}{q^2 - m_c^2 c^2} (2\pi)^4 \delta^4(p_2 - q - p_3) (2\pi)^4 \delta^4(p_1 + q - p_4) \frac{d^4 q}{(2\pi)^4}$$

$$\Downarrow$$

$$(-ig)^2 \frac{i}{(p_4 - p_1) - m_c^2 c^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$\Downarrow$$

$$M = \frac{g^2}{(p_4 - p_1) - m_c^2 c^2}$$